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# Classical Chaos and Quantum Level Density of an Anharmonic Oscillator

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The Liapunov exponents of two-dimension anharmonic oscillator systems are studied through numerical calculations. The result shows that the systems consist of regular and irregular regions in phase space in the classical limit. The corresponding quantum systems are investigated. The distribition $P(s)$ of spacings between adjacent energy levels indicates a corresponding transition from Poisson-like distribution to Wigner-like distribution. $P(s)$ is dependent on the total irregular fraction of phase space.

## 1. INTRODUCTION

The onset of chaos in nonintegrable conservative classical systems of few degrees of freedom is by now well characterized (Casati and Ford, 1979). There has been considerable interest in determining whether the corresponding quantum systems carry a signature of classical chaos. The statistical properties of the spectra of quantum systems have been found to be a significant measure for the degree of integrability of a chaotic system in its classical limit. Several useful statistical measures have been introduced. The most important ones are the distribution of energy level spacings and $\Delta_3$ statistic measuring short- and long-range correlations of the spectral sequence, respectively (Berry and Tabor, 1977; Casati, 1985; Seligman and Verbaarschot, 1985). Strongly chaotic systems in quantum mechanics give rise to the spectral statistics of the Gaussian orthogonal ensemble of matrices.

It has been argued by Percival (1978) that in the semiclassical limit a spectrum should consist of regular and irregular parts that are associated with the classical regular and irregular regions in phase space. The level spacing distribution of a completely regular system assumes a Poisson-like form in the semiclassical limit (Berry and Tabor, 1977), and a completely

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irregular system shows a Wigner-like level spacing distribution. A mixture of these two distributions has been obtained (Berry and Robnik, 1984) for systems in which the corresponding classical system has regions of both regularity and irregularity. The distribution is

$$P(q, s)=\exp[-(1-q)s-\tfrac{1}{4}\pi q^2 s^2][1-q^2+\tfrac{1}{2}\pi q^3 s-(1-q^2)R(qs)] \quad (1.1)$$

where

$$R(z)=1-\exp(\tfrac{1}{4}\pi z^2)\,\mathrm{erfc}(\tfrac{1}{2}\pi^{1/2}z)$$

There is a parameter $q\in[0,1]$. When $q=1.0$ the distribution is the Wigner distribution $P(s)=\frac{1}{2}s\exp(-\frac{1}{4}\pi s^2)$; for $q=0.0$ it is the Poisson distribution, $P(s)=\exp(-s)$. It is of interest to obtain the parameter $q$. For a Hamiltonian system of two coupled quartic oscillators

$$H=\tfrac{1}{2}(P_x^2+P_y^2+X^2+Y^2)+4kX^2Y^2 \quad (1.2)$$

the quantum mechanical values $q_{\mathrm{qm}}$ are obtained by a least-squares fit of the nearest neighbor energy level spacing distribution (Zimmermann *et al.*, 1986). From the classical mechanics point of view, it is the Liouville measure of the irregular region divided by the measure of the energy shell (Meyer *et al.*, 1984)

$$q=\left[\int d\gamma\,\delta(H(\gamma)-E)\chi(\gamma)\right]\Big/\left[\int d\gamma\,\delta(H(\gamma)-E)\right] \quad (1.3)$$

For some Hamiltonian system it is convenient to get the fraction $q$ with a method suggested by Meyer (1986).

In the present paper we study the Hamiltonian system

$$H=\tfrac{1}{2}(P_x^2+P_y^2+X^2+Y^2)+\frac{\alpha'}{4}(aX^4+bY^4+2cX^2Y^2) \quad (1.4)$$

With some parameters $a$, $b$, and $c$, we calculate the nearest energy level spacing distribution to get $q_{\mathrm{qm}}$ and study the Liapunov exponents for its corresponding classical system to obtain $q_{\mathrm{cl}}$.

In Section 2 we give Liapunov exponents for the Hamiltonian system and the fraction of the irregular region in total phase space. In Section 3 we study the nearest neighbor level spacing distribution of the system and $q_{\mathrm{qm}}$ from the distribution. In Section 4 we discuss the numerical results.

## 2. LIAPUNOV EXPONENTS OF THE HAMILTONIAN SYSTEM

The Liapunov exponent is the speed with which neighboring trajectories separate exponentially. The theory of the Liapunov exponent for Hamiltonian system was introduced by H. D. Meyer.

For a classical Hamiltonian system of $F$ degrees of freedom, the points in $2F$-dimensional phase space are

$$\gamma(x_1, x_2, \ldots, x_F, p_1, p_2, \ldots, p_F)$$

The stability matrix for trajectory $\gamma(t)$ with initial conditions $\gamma(0)$ is defined as

$$M_{ij}[\gamma(0), t] = \frac{\partial \gamma_i(t)}{\partial \gamma_j(0)} \tag{2.1}$$

and the Liapunov function as

$$\lambda(t) = t^{-1} \ln[(2F)^{-1/2} \|M(t)\|] \tag{2.2}$$

where $\|\cdot\|$ denotes the Euclidean norm. The maximal Liapunov exponent is usually of particular interest. For instance, the set of Liapunov exponents of the systems with two degrees of freedom is completely determined by the maximal one,

$$(2F)^{-1/2} d_F \leq (2F)^{-1/2} \|M\| \leq d_F \tag{2.3}$$

where $d_F$ is the maximal diagonal element of the matrix $M$,

$$\lambda = \lim_{t\to\infty} \lambda(t) = \lambda_F$$

The time evolution of the stability matrix is as follows:

$$M[\gamma(0), t+h] = M[\gamma(t), h] M[\gamma(0), t] \tag{2.4}$$

$$M[\gamma(t), h]_{ij} = \frac{\partial \gamma_i(t+h)}{\partial \gamma_j(t)} \tag{2.5}$$

$\gamma_i$ is given by the Taylor series

$$\gamma_i(t_0 + h) = \gamma_i(t_0) + \sum_{n=1}^{N} \frac{h^n}{n!} \frac{d^n \gamma_i}{dt^n} \tag{2.6}$$

For the Hamiltonian system

$$H = \tfrac{1}{2}(p_x^2 + p_y^2 + x^2 + y^2) + \frac{\alpha'}{4}(ax^4 + by^4 + 2cx^2y^2) \tag{2.7}$$

Transform the variables in (2.7) into dimensionless variables

$$x \to x/s; \quad y \to y/s; \quad P_x \to P_x/s; \quad P_y \to P_y/s; \quad H \to H/E; \quad \alpha' \to \alpha \tag{2.8}$$

We get each term in the Taylor series; then $M[\gamma(0), t]$ and the Liapunov function are calculated. Figure 1 shows the Liapunov function of the trajectory started at the point ($x = 0.0$, $y = 0.2607$, $P_y = 0.7211$) for the

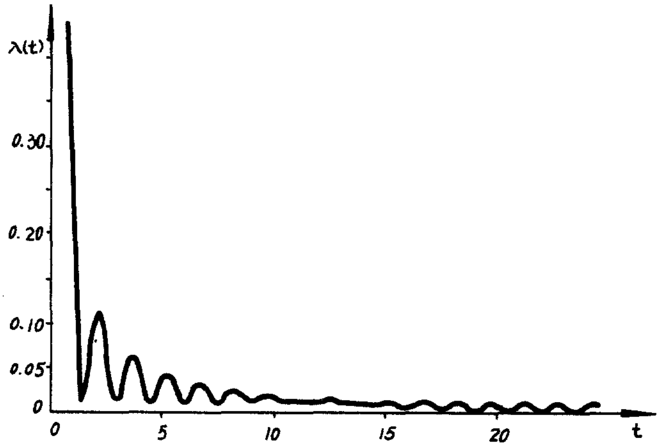


**Fig. 1.** The Liapunov function of an irregular trajectory. The trajectory was started at $x = 0$, $y = 0.2607$ and $p_y = 0.7211$ on the energy shell $E = 25.0$ for the Hamiltonian of (2.7) with the parameters $a = 3.8$, $b = 0.7$, $c = 2.4$, and $\alpha = 0.088$.

Hamiltonian system with parameters $a = 3.8$, $b = 0.7$, $c = 2.4$, $\alpha = 0.088$, and $E = 25.0$. For integrable systems the Liapunov function will take the form $(2t)^{-1}\ln(1+\beta^2 t^2)$. The Hamiltonian system of (2.7) with $a = 1.0$, $b = 1.0$, $c = 1.0$, and $\alpha = 0.04$ is an integrable system. We calculate the Liapunov function of a trajectory which started from the point ($x = 0.0$, $y = 0.1395$, $P_Y = 0.2828$) on the energy shell $E = 1.0$. When $t$ is large enough, $\lambda(\gamma, t)$ is around 0.000035; it is close to zero (see Figure 2).

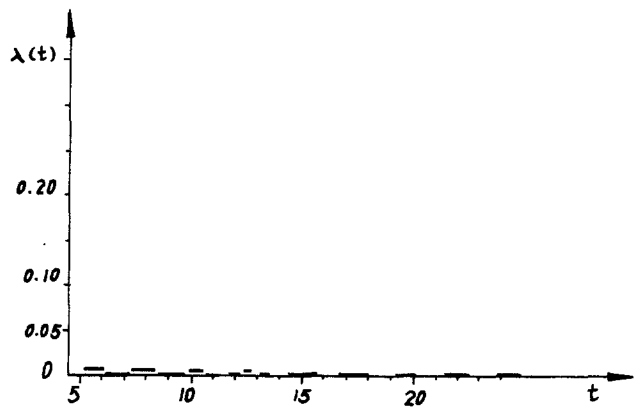


**Fig. 2.** The Liapunov function of a regular trajectory started at $x = 0$, $y = 0.1395$ and $p_y = 0.2828$ on the energy shell $E = 1.0$ for the Hamiltonian of (2.7) with parameters $a = b = c = 1.0$ and $\alpha = 0.04$.

A system is in general not simple regular or irregular, but regular in some regions of phase space and irregular in others. The phase space $\Gamma$ decomposes into a regular part $\Gamma_R$ and an irregular part $\Gamma_I$, that is, $\Gamma = \Gamma_R + \Gamma_I$. A measure $q$ is defined as the relative weight of the irregular part of the energy shell, $q = |\Gamma_I \cap \Gamma_E| / |\Gamma_E|$. The symbol $|A|$ denotes the volume of $A$. The characteristic function $\chi$ for the point $\gamma$ in phase space is

$$\chi(\gamma) = \begin{cases} 1 & \text{if} \quad \lambda(\gamma) > 0 \\ 0 & \text{if} \quad \lambda(\gamma) = 0 \end{cases} \tag{2.9}$$

where $\lambda(\gamma)$ denotes the maximal Liapunov exponent of a trajectory started at $\gamma$. Therefore

$$q = \frac{\int d\gamma\, \chi(\gamma) \delta[H(\gamma) - E]}{\int d\gamma\, \delta[H(\gamma) - E]} \tag{2.10}$$

It is convenient to consider the Poincare surface of the section $S_E$,

$$S_E = \{(y, p_y)^+ | \exists p_x > 0: \quad H(0, y, p_x, P_y) = E\} \tag{2.11}$$

Corresponding to $S_E$, $q_s$ is defined by

$$q_s = \left[ \int_{S_E} dy\, dP_y\, \chi(\gamma) \right] \Big/ \int_{S_E} dy\, dP_y \tag{2.12}$$

For the Hamiltonian system (2.7), $q_s$ is evaluated numerically. We divided the surface of section into small rectangular cells on the energy shell $E = 1.0$,

$$C_{ij} = \{(y, P_y)^+ | (i-1)\Delta y < y \le i\Delta y, (j-1)\Delta P_y < P_y \le j\Delta P_y\} \tag{2.13}$$

where

$$y = y_{\max}/50 \qquad P_y = P_{y,\max}/50 \tag{2.13}$$

We calculate the Liapunov exponents of the trajectories started from points in every small cell. We compare the Liapunov exponents with some threshold $\lambda_s$, which is chosen from an integrable Hamiltonian system such as the Hamiltonian in (2.7) with parameters $a = b = c = 1.0$, $\alpha = 0.04$. We take $\lambda_s = 0.000035$. If $\lambda \ge \lambda_s$, then $\chi(\gamma) = 1$; otherwise, $\chi(\gamma) = 0$. According to (2.12), we have $q_s = 0.941$ for the Hamiltonian system in (2.7) with $a = 2.5$, $b = 1.0$, $c = 2.0$, $\alpha = 0.04$, and $E = 1.0$. It is the result from the classical trajectory. When the parameters are $a = 1.02$, $b = 0.97$, $c = 0.98$, $E = 1.0$, and $\alpha = 0.04$ we have $q_s = 0.187$.

## 3. QUANTUM ENERGY SPECTRUM OF HAMILTONIAN SYSTEM

We solve the Hamiltonian system given in (2.7) by a perturbation method. Let

$$x = \frac{1}{\sqrt{2}}(a_1^+ + a_1), \qquad y = \frac{1}{\sqrt{2}}(a_2^+ + a_2)$$

With two-dimensional harmonic oscillator eigenvectors the first-order correction to the energy is calculated through a degenerate perturbation method. Therefore we get 4560 energy levels. We consider the lower 2500 energy levels to study the statistical properties of the spectrum. The nearest energy level spacing distribution for a chaotic system is the Wigner-like distribution, and for an integrable system the distribution is the Poisson distribution. Recently, special attention has been paid to systems that classically show a transition between the two limit cases of totally regular and chaotic behavior. These systems are intermediate systems. The spectrum for these systems is generated by a statistically independent superposition of two kinds of distribution, as in (1.1).

A similar distribution for intermediate systems is used in the present paper to compare with the numerical result

$$P(q, s) = \left(\frac{\pi q^2}{2}\frac{s}{D^2} + \frac{1-q}{D}\right)\exp\left[-(1-q)\frac{s}{D} - \frac{\pi q^2 s^2}{4D^2}\right] \tag{3.1}$$

where $s$ is the level spacing, $D$ is a parameter which is concerned with mean level spacing, and $q$ is a parameter. We get the value of $q$ from the

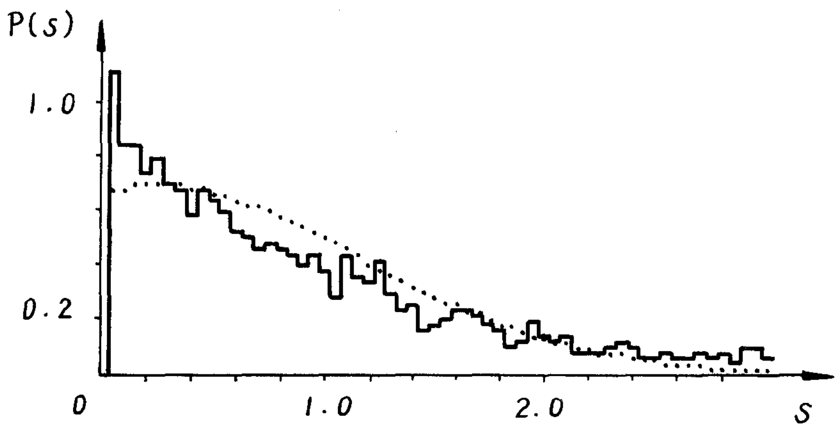


**Fig. 3.** Level spacing histogram for the Hamiltonian with parameters $a = 2.21$, $b = 0.75$, $c = 1.50$, and $\alpha = 0.04$. The curve is $P(s, q_{\mathrm{qm}})$ with $q_{\mathrm{qm}} = 0.50$.

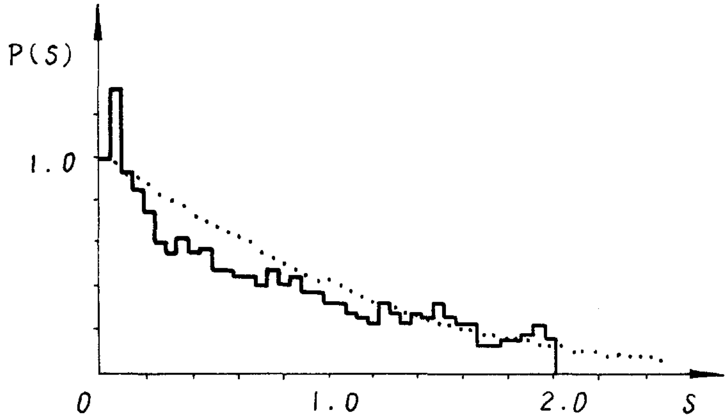


**Fig. 4.** Level spacing histogram for the Hamiltonian with parameters $a = 0.95$, $b = 1.57$, $c = 1.85$, and $\alpha = 0.04$. The curve is $P(s, q_{qm})$ with $q_{qm} = 0.30$.

numerical result by a fit of the function $P(q, s)$ in the level spacing histogram for the distribution found by numerical calculation. The quantum Hamiltonian system of (2.7) with parameters $a = 2.21$, $b = 0.75$, $c = 1.50$, and $\alpha = 0.04$ is studied. The nearest energy level spacing distribution is shown in Figure 3. The histogram is from numerical results, the curve is the distribution (3.1), with $q = 0.50$, $D = 0.75$. Here $q$ is obtained from the quantum energy spectrum. We denote it as $q_{qm}$, which is less than the classical result for the chaotic phase-space fraction $q_{cl} = 0.90$. The dotted line in Figure 5 shows the $\Delta_3$ statistic for this case, while the straight line is $\Delta_3(L) = L/15$ and the curve is $\Delta_3(L) = (1/\pi^2) \ln L - 0.007$.

When the parameters of the system are $a = 0.95$, $b = 1.57$, $c = 1.85$, and $\alpha = 0.04$, we have the histogram of the nearest energy level for this system shown in Figure 4, which compares it with the curve of $P(q, s)$ in (3.1) with

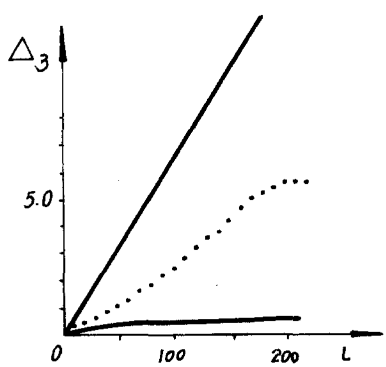


**Fig. 5.** $\Delta_3(L)$ for the Hamiltonian in Figure 3. Solid lines are $L/15$ and the GOE prediction.

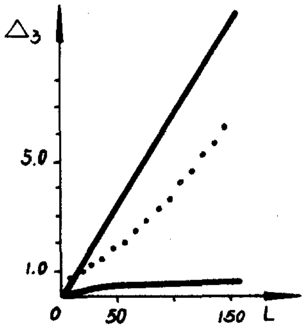


**Fig. 6.** $\Delta_3(L)$ for the Hamiltonian in Figure 4. Solid lines are $L/15$ and the GOE prediction.

$q = 0.30$, $D = 0.70$. The fitted $q_{qm}$ is less than the $q_{cl} = 0.88$. The $\Delta_3$ statistic for this case is given in Figure 6. A similar case is studied, as for the system with parameters $a = 2.5$, $b = 1.0$, $c = 2.0$, and $\alpha = 0.04$ we have $q_{qm} = 0.87$ and $q_{cl} = 0.941$.

## 4 DISCUSSION

When the fraction of phase space of the classical Hamiltonian system is filled with irregular trajectories, its corresponding quantum system shows the property that the nearest energy level distribution is a superposition of a Wigner distribution and a Poisson distribution with relative weights $q_{qm}$ and $1 - q_{qm}$, respectively. The value of $q_{qm}$ given in the energy level spacing distribution and $q_{cl}$ do not coincide. For the system with large $q_{cl}$ the $q_{qm}$ of the corresponding system is less than $q_{cl}$, while for smalll $q_{cl}$ the quantum value $q_{qm}$ is larger than $q_{cl}$.

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